Comparing the Spanish and the discriminatory auction formats: A discrete model with private information*

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Abstract: The Spanish Treasury is the only Treasury in the world that uses a hybrid system of discriminatory and uniform price auctions to sell government debt: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids (WAP), and pay the WAP otherwise. Following Gordy (1996), we model the Spanish auction as a common value auction of multiple units with private information, allowing for multiple bids. Numerical analysis shows that bidders spread their bids more in the Spanish than in the discriminatory auction and bid higher for the first unit, and that the expected seller's revenue is higher in the Spanish than in the discriminatory auction within a reasonable set of parameter values. Keywords: Auctions, Bidding, Simulations, Treasury auctions, Multi-unit auctions. JEL classification: D44.

1 Introduction

Auctions are used worldwide to sell government debt. Treasuries apply mainly two auction formats, discriminatory and uniform, favoring the former.[1] In a discriminatory auction, winning bidders pay their bid price for each unit. In

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[1] According to Bartolini and Cotarelli (1997) in 1994, 39 out of the 42 countries that used auctions to sell Treasury securities used discriminatory auctions and only 2 uniform auctions, Spain being the only country that used a different format. However, Treasuries in the U.S., Mexico and Italy have lately changed to the uniform format.

a uniform price auction, all winning bidders pay the same price for each unit, the minimum accepted price. The Spanish Treasury is the only one that uses a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids, whereas they pay the weighted average price of winning bids otherwise.

Most of the discussion on Treasury auction design focuses on the choice between discriminatory and uniform auctions, addressing the issue of which auction format yields the seller higher revenue, but analyses of the Spanish auction are scarce. This paper analyzes the Spanish format, and compares equilibrium strategies and seller's revenue between the Spanish and the discriminatory format. Given that the total supply of securities sold in Treasury auctions is very large, even a small revenue advantage of one format over the other can substantially reduce the government's refinincing costs, and is worthwhile investigating.

We present a common value model of Treasury auctions that explicitly account for multiple units of a good on sale and bidders that demand multiple units. Many authors that study multiple-unit auctions where bidders demand more than one unit follow the "share auctions" approach, proposed by Wilson (1979), where the good is assumed to be perfectly divisible, price is a continuos variable and a bid is a smooth demand schedule. In this paper we depart from this continuous approximation: we use a discrete model both on quantities and prices, since this better corresponds to how Treasury auctions work in practice. For example, in Treasury bill auctions in Spain, the quantity for a single bid must be a multiple of 1000 euros, and prices must be in increments of 5 cents of an euro.[2] These type of restrictions are common to most Treasury auctions.

In our model, bidders have asymmetric information about the value of the good. This assumption is reasonable for Treasury auctions, given that bidders are heterogeneous: both dealers and smaller investors bid in the auction. For example, dealers may acquire private information participating in the when-issued market, in which they may either take long or short positions in the Treasury security to be auctioned: as Nyborg and Sundaresan (1996) point out, if a dealer receives a few large orders from some institutional customers, only that dealer has that information. Also, institutions often place bids in the auction through dealers, who obtain in this way private information about the aggressiveness of bidding (Sirri, 1993).

Considering private information in a multiple units model increases the dimensionality of the available information, given the addition of the multiple signals that each bidder can receives, and therefore increases the complexity of the model.[3] In addition, with the Spanish format the price that a bidder pays for certain units depends on the bids of all winning bidders, including his own

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[2] See http://www.tesoro.es

[3] See, for example, Krishna (2002).

bids. This fact increases the bidders' strategic considerations with respect to discriminatory and uniform auctions. We use numerical simulations to explore the equilibria of the model, since we are not able to solve it analytically, even for the simplest case. For reasonable combinations of parameters, we find all symmetric Bayesian Nash equilibria, if they exist, for the Spanish and the discriminatory auction, and compare equilibria in terms of the use of bid spread and expected seller's revenue.

This paper relates to a growing theoretical literature on multiple-unit and multiple-demand auctions. An abundant literature exits analyzing the auctioning of a single, indivisible item; the analysis has been extended to settings with multiple units in which each bidder has a taste for only one item.[4] However, general results for auctions of multiple units in which bidders desire multiple items remain elusive.

The most closely related paper to ours is Gordy (1996). He uses numerical simulations to analyze the discriminatory auction, and finds evidence that supports the conjecture that multiple bidding can be used to hedge against uncertainty and the winner's curse, when bidders are risk averse. Our paper builds on his model extending the analysis to the Spanish auction format.

Other papers that also use a discrete model to analyze multiple units with multiple bids auctions, are Engelbrecht-Wiggans and Kahn (1998a and1998b) for the independent private values case with risk neutral bidders, and Kremer and Nyborg (2004a), for the common value with symmetric information and risk neutral bidders case. Engelbrecht-Wiggans and Kahn (1998b) consider the auction of two units of an indivisible item with a discriminatory format, and establish that with positive probability, a bidder will bid the same for both units even though he values them differently. Engelbrecht-Wiggans and Kahn (1998a) present a similar model for the uniform format, with M units of a good to be auctioned, $M \geq 2$, when each bidder wishes to purchase two units of the good, and find that in equilibrium there is bid spread, and bidders' lower bids are shaded strictly below their valuation of the good. Kremer and Nyborg (2004) analyze the uniform auction, and show that when there is both quantity and price discreteness, as in our model, there are equilibria with underpricing, that can be arbitrarily small by choosing a sufficiently small price tick size and a sufficiently large quantity multiple.

Wang and Zender (2002) consider multiple units auctions with common values and private information as we do, but they use a continuous-unit model. They obtain the ordinary differential equations that characterize the equilibrium strategies for the uniform and the discriminatory auction, using a generic

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[4] See for example Milgrom and Weber (1982) for a characterization of auctions of one indivisible unit, and Harris and Raviv (1981) and Weber (1983) for a characterization of auctions on multiple units, where bidders only demand one unit.

probability distribution for private information.[5] However, as they point out, "because of the complex inference problem embedded in the derivation of the equilibrium bid schedules, obtaining explicit solutions to the general problem is nontrivial". They just present a parametric example for the uniform auction. Since the equilibrium bid schedules that they obtain for the uniform auction do not depend on the distributional properties of the signals, they do not need to assume a specific distribution for the signal.[6]

Little attention has been paid to the Spanish format. Salinas (1990) presents a private-value single-unit demand model and Mazón and Nuñez (1999) restrict the analysis to a case in which only one bid is permitted, a bid being a price-quantity pair. Álvarez et al. (2003) solve a continuos model with common value and no private information. Finally, Abbink et al. (2002) report an experiment that compares the discriminatory, uniform and Spanish designs within a model that is more restrictive than ours regarding the probability distribution function that drives the bidder's valuations.[7] In line with our results, they find significantly higher seller's revenue with the Spanish than with the discriminatory format.

Our paper extends the existing literature because it presents for the Spanish auction a multiple units and multiple bids model with common value and private information, in which both the sets of possible prices and quantities that bidders can bid from are discrete; we think that this model captures important characteristics of Treasury auctions. We solve the model numerically, using an algorithm to look for the equilibria that goes beyond Gordy's (1996): whereas he does a random search for equilibria in the strategy space, we use an algorithm that finds all (if any) symmetric Bayesian Nash equilibria. This exhaustive search is important for the model considered since, as we show, multiple equilibria arise for a wide range of parameter values.

Our main findings can be summarized as follows. Firstly, bidders spread their bids more in the Spanish than in the discriminatory auction. There are two contributing factors. On the one hand, bidders bid more aggressively on the first unit in the Spanish than in the discriminatory auction, since they have a lower expected cost of doing so: if they win with the highest bids, they only pay the weighted average price of winning bids instead of their bids, as they do in the discriminatory auction. On the other hand, in most equilibria the bid on the second unit is lower in the Spanish than in the discriminatory auction, since bidders have incentives to lower their bids on later units, as they might determine the price they pay for earlier units. For both the Spanish and the

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[5] And restricting attention to equilibria bid schedules that are additively separable in price and a bidder's private signal.

[6] Note, however, that they do not provide an example for the discriminatory auction, for which the equilibrium bid schedules that they obtain does depend on the distributional properties of the signals.

[7] They assume that both the value of the good and the signals are uniformly distributed.

discriminatory auction, bid spread increases as the parameter of risk aversion, the uncertainty about the value of the good, or the number of buyer increases. Secondly, the ranking of the Spanish and the discriminatory auctions in terms of expected seller's revenue varies with the value of the parameters of the model, and when there are multiple equilibria might depend upon which of the equilibria is considered. Within the set of parameters considered, the Spanish outperforms the discriminatory auction in terms of expected seller's revenue in a number of cases, specially when competition is high, and when risk aversion and/or the uncertainty about the value of the good are low.

The paper is organized as follows. Section 2 presents the model and methodology, Section 3 presents the results and Section 4 concludes.

2 Model and methodology

2.1 Model

We adapt Gordy's (1996) model to the Spanish auction. Two indivisible and identical units are for sale, and $I \geq 2$ bidders compete for them. Each bidder submits two sealed bids, specifying a price, but not a particular unit.

The true value of each unit of the good for sale, v, is unknown to the bidders at the time of bidding. Let $F(v)$ be the prior distribution of v. We assume that $F(v)$ is public information and is $Beta\left(\alpha\mu, \alpha(1-\mu)\right)$. This distribution has mean μ and a variance decreasing in α.[8] Additionally, the bidders have private information on v: each bidder observes privately a signal from the finite set $X := \{0, 1, ..., K\}$, with $K > 0$. The probability distribution of the signal conditional on v is assumed to be $Binomial\,(K, v)$. Bidders combine public information, the prior on v, and private information, the signal received, using Bayes' rule. Notice that the posterior distribution of v is $Beta\left(x + \alpha\mu, K - x + \alpha(1-\mu)\right)$, where x is the signal that the bidder has received ($x \in X$).[9] Particularly, the posterior expectation of v is $E\{v/x\} = \delta(x/K) + (1-\delta)\mu$, where $\delta = (1 + \alpha/K)^{-1}$.

As mentioned in the Introduction, Treasury auctions in Spain (and in most countries) have restrictions on the set of price bids permitted. We allow for a finite set of prices: $\Lambda := \{0, 1/\lambda, 2/\lambda, ..., 1\}$, where λ is some positive integer.

We assume that bidders are risk averse[10] and that they have a constant absolute risk aversion (CARA) utility function, $U(z) = -\exp(-\rho z)$, where

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[8] The beta distribution, of which the uniform distribution is a special case, can take on a variety of unimodal and bimodal forms with support in $[0,1]$, and is well suited for modeling the distribution of the true value of the good.

[9] See DeGroot (1970).

[10] Evidence from Portuguese Treasury auctions reported in Gordy (1999) suggest that bidders are risk averse.

$\rho > 0$ is the coefficient of absolute risk aversion, common to all bidders, and z is the profit obtained by the bidder on the auction, that depends on the auction format.

We consider two auction formats, the Spanish and the discriminatory. Under both formats, the units are awarded to the highest bids, and if a tie occurs, there is a randomization among the tied bids. Profits to bidders from the auction are calculated as the valuation minus payments. For both auction formats, the valuation is equal to $2v$ if a bidder wins two units and the value of one unit is v; and is equal to v if he only wins one unit. Payments depend on the auction format. In the Spanish auction, winning bids pay the bid price for each unit if this is lower than the weighted average price of winning bids (WAP), and pay the WAP if the bid price is equal to or higher than the WAP. In the discriminatory auction, winning bids pay the bid price for that particular unit. For example, assume that there are two bidders, A and B, bidding $\left(\frac{3}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, \frac{1}{5}\right)$, respectively. Winning bids are $\frac{3}{5}$ and $\frac{2}{5}$, and each bidder gets one unit in both auction formats. Bidder A pays $\left(\frac{3}{5} + \frac{2}{5}\right)/2 = \frac{1}{2}$ in the Spanish auction and $\frac{3}{5}$ in the discriminatory auction; bidder B pays $\frac{2}{5}$ in both auction formats.

The model is a simultaneous game of incomplete information. We consider pure strategy Bayesian Nash equilibrium. A strategy or a decision rule for bidder i is a function, $\mathbf{s}_i$, from the set of signals to the Cartesian product of the set of prices, $\mathbf{s}_i : X \rightarrow \Lambda \times \Lambda$, which gives the bidder's pair of bids for each possible realization of the signal. Let $\mathbf{s}_{-i}(\mathbf{x}_{-i}) \in (\Lambda \times \Lambda)^{I-1}$ be a pair of bids for all bidders but i given $\mathbf{x}_{-i}$, where $\mathbf{x}_{-i} \in X^{I-1}$ is the vector of signals for all bidders but i. Since bidder i's profit only depends on $(\mathbf{s}_i(x_i), \mathbf{s}_{-i}(\mathbf{x}_{-i}))$, we can write his utility as $U_i(\mathbf{s}_i(x_i), \mathbf{s}_{-i}(\mathbf{x}_{-i}))$. A profile of decision rules $(\mathbf{s}_1, ..., \mathbf{s}_I)$ is a Bayesian Nash equilibrium if and only if

$$E\left\{U_i\left(\mathbf{s}_i(x_i), \mathbf{s}_{-i}(\mathbf{x}_{-i})\right) \ / \ x_i\right\} \geq E\left\{U_i\left(\widehat{\mathbf{s}}_i, \mathbf{s}_{-i}(\mathbf{x}_{-i})\right) \ / \ x_i\right\}$$

for all i, $x_i \in X$ and $\widehat{\mathbf{s}}_i \in \Lambda \times \Lambda$, where the expectation is taken over v and $\mathbf{x}_{-i}$ and is conditional on x_i. That is, for all bidders, the bid pair for any signal maximizes the bidder's expected utility given the strategies of all other bidders. We only consider symmetric equilibria, $(\mathbf{s}_1, ..., \mathbf{s}_I) = (\mathbf{s}, ..., \mathbf{s})$, that is, equilibria in which all bidders play the same strategy. In what follows we refer to symmetric Bayesian Nash equilibrium simply as equilibrium.

The model is not analytically solvable to the best of our knowledge, so we proceed numerically. The two main questions of the implementation of the numerical solution are the choice of the parameter values and the design of the algorithm to find equilibria. In the remainder of this section we discuss both questions.

2.2 Choice of parameter values

The vector of parameters of the model is $(\mu, \alpha, \rho, I, \lambda, K)$.

We have kept μ, the *a priori* expected value of the value of the good for sale, equal to $0,75$ throughout all the combinations of parameter values considered. Note that μ must lie in $(0,1)$, and that if μ were too small, the *a posteriori* expected value of v would be too small for bidders to bid strictly positive bids for every possible signal.

For the parameter of accuracy of prior information on the value of the good, α, we have selected values in $\{4,6,8,10,12,14,16\}$.[11]

For the risk aversion parameter, ρ, we have selected values in $\{1,2,...,5\}$, motivated by the macroeconomic literature.[12]

We have used data on Spanish Treasury auctions to identify reasonable values for the number of bidders, I.[13] Unfortunately, there is no public data on individual bidding, but there is aggregate data on competitive demand and total supply.[14],[15] We consider auctions of one-year Treasury bills that took place between May 1993 and October 2004; there were 246 auctions in that period. Figure 1 presents the evolution of the ratio of total supply to competitive demand over time. In that period the ratio was relatively stationary, with an average (across auctions) of $0,55$. In our model, the supply is 2 units, and we only consider competitive bidders, denoted by I; the previous ratio is $0,5$ for $I=2$ and $0,33$ for $I=3$. Thus, we have considered $I=2$ and $I=3$. In addition, we find that for a larger number of bidders and only two units for sale, the multiplicity of equilibria increases dramatically: as the number of bidders increases, the probability of winning with the lower bid decreases, and there are multiple equilibria with the same high bid that only differ on the lower bid.

The remaining parameters, λ, where $\lambda+1$ is the number of prices the bidders can choose from, and K, where $K+1$ is the number of possible signals, affect the computation time. To keep the computation time within an acceptable limit, we have considered λ and K in $\{3,4,5,6\}$.

We have looked for equilibria for the Spanish and discriminatory formats for all the combination of parameter values within the range of values indicated

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[11] We have look for equilibria with values of α lower than 4, but a problem of existence of equilibrium arises, probably because the variance of the value of the good is too large. And we have tried higher values than 16, but a technical problem arises: when we compute the expected utility of following a strategy, we must average across all possible vectors of rivals' signals, using the probability of each such vector conditional on each possible signal. If α is "large", some signal realizations are very unlikely, what implies that for those realizations the conditional probability is of an order of magnitude of $10^{-8}/10^{-7}$, or, roughly, undetermined. Clearly, what is a "large" α does depend on I and K.

[12] See, for example, Aghion and Howitt (1998).

[13] The data can be downloaded from http://www.bde.es

[14] The Spanish Treasury allows both competitive bids, where bidders specify both a bid price and a bided quantity for that particular price, and no competitive bids where bidders do not specify a bid price, but only the quantity demanded. No competitive bids are a small percentage of total demand.

[15] Other data avalaible are aggregate non competitive demand, the weithted average price of winning bids, the price of the last accepted bid and the price of the first non accepted bid.

above, except for those combinations in which simultaneously $\lambda = 6$ and $K = 6$.[16] That makes a total of 1050 combinations of parameter values.[17]

2.3 Algorithm

Given a combination of parameter values, there are two questions to address when searching for equilibria. First, how to explore the strategy set, and second, how to check whether a given strategy is an equilibrium.

Respect to the first question, denote by Γ the set of all possible functions $\mathbf{s} : X \to \Lambda \times \Lambda$. As both X and Λ are finite, so is Γ. In order to shorten computation time, we consider only strategies in which the high bid, $s^1(x_i)$, is non-decreasing in the observed signal.[18],[19] Denote by $\widehat{\Gamma}$ the subset of Γ containing such strategies. We list all the elements of $\widehat{\Gamma}$ in some arbitrary order, and for every strategy $\mathbf{s} \in \widehat{\Gamma}$ we check whether $(\mathbf{s}, \mathbf{s}, ..., \mathbf{s})$ is an equilibrium. Thus we find all (if any) equilibrium strategies for each combination of parameter values considered. In contrast, Gordy (1996) follows a *tâtonnement* approach: he starts with $\mathbf{s} \in \Gamma$ chosen at random, assumed to be played by $I - 1$ bidders, and computes the best response to it (in Γ) by the remaining bidder, say $f(\mathbf{s})$. If $f(\mathbf{s}) = \mathbf{s}$, then $(\mathbf{s}, \mathbf{s}, ..., \mathbf{s})$ is an equilibrium and another element in Γ is selected as a candidate at random. If $f(\mathbf{s}) \neq \mathbf{s}$, then the best response to $f(\mathbf{s})$ is computed, $f(f(\mathbf{s}))$, and the process is repeated until some strategy $\mathbf{s}'$ is the best response to itself ($f(\mathbf{s}') = \mathbf{s}'$). Note that Gordy's exploration of Γ might enter an infinite loop (i.e.: $f(f(\mathbf{s})) = \mathbf{s}$ and $f(\mathbf{s}) \neq \mathbf{s}$), and that since the element in Γ selected after finding an equilibrium is chosen at random, the exploration of Γ is random.

Respect to the second question, how to check whether a given strategy is an equilibrium, we essentially extend Gordy (1996)'s analysis of the discriminatory format to the Spanish format. In the Appendix we present some results that ease the computation of the bidder's expected utility, together with some auxiliary references.

Given a vector of parameters, the computation time of checking for equilibria only depends on I, λ and.K. More specifically, I and K determine jointly the number of possible vectors of rivals' signals, which in turn determines the computation time required to evaluate whether a strategy is an equilibrium, whereas λ and K determine jointly the number of strategies, that is, the cardinality of Γ.

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[16] For $\lambda = 6$ and $K = 6$, with two bidders, computation time for equilibria for a given combination of parameters is 1298 minutes

[17] 7 values of α, 5 values of ρ, 2 values of I, 4 values of λ and 4 values of K ($7 \times 5 \times 2 \times 4 \times 4 = 1120$) minus 70 combinations for $\lambda = K = 6$.

[18] Given any signal $x \in X$, let the corresponding bid pair be $\mathbf{s}(x) = (s_1(x), s_2(x)) \in \Lambda \times \Lambda$, in which, without loss of generality, we take $s_1(x) \geq s_2(x)$. We refer to $s_1(x)$ and $s_2(x)$ as the high and the low bid, respectively.

[19] Gordy (1996) looks for equilibria within Γ and in every equilibria that he obtains for the discriminatory auction, the high bid is non-decreasing in the signal, as we are assuming.

The number of possible vector of rivals' signals is approximately $(I-1)^K$and the cardinality of Γ is approximately K^λ. Table 1 presents the computation time for different values of those parameters. We have used TurboPascal on a Pentium 4 at 3.20GHz.[20]

3 Results

We start this Section with an example. Figure 2 shows a strategy that is an equilibrium for the Spanish format. The parameter values are $(\alpha, \rho, I, \lambda, K) = (4, 1, 2, 5, 4)$. The vector of possible signals is represented on the horizontal axis, and the vector of possible prices on the vertical axis. Since λ equals 5, permitted bids are $\left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$, and since K is equal to 4, there are 5 possible signals, $\{0, 1, 2, 3, 4\}$. For each signal, we plot the high (circle) and low (square) bid.

When considering the equilibria, an initial question that arises is that of existence and uniqueness. We check for equilibria for 1050 combinations of parameters, and find at least one equilibrium for 627 of those parameter combinations for the Spanish auction, and for 1033 for the discriminatory auction; i.e., there exists at least one equilibrium for 59,7% of the parameter combinations considered for the Spanish auction, and for 98,4% for the discriminatory auction. In addition, the equilibrium is unique in 69,9% of the parameter combinations for which there is at least an equilibrium for the Spanish auction, and in 39,7% for the discriminatory auction.

In the literature, the problem of existence of equilibria has been linked to the continuity of the strategy space. For example, Haller and Lengwiler (1998) prove the existence of equilibrium in a multi-unit discriminatory auction in a discrete model and conclude that non-existence is an artifact of the Menezes and Monteiro (1995) model, which considers prices and quantities as continuous variables. McAdams (2002) confirms this idea for the uniform auction. In contrast, our paper presents a discrete model, with common value and private information in which there is no symmetric equilibria for certain parameter values; the problem is more acute for the Spanish than for the discriminatory auction.

The non-uniqueness of equilibria is usually linked to the uniform auction format. For example, Wang and Zender (2002), in a continuous-unit model with symmetric bidder information, find a continuum of equilibria for the uniform auction, but a unique weakly downward sloping equilibrium for the discriminatory auction. In contrast, our model presents multiple equilibria for the discriminatory auction in 60,3% of the parameter combinations considered for

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[20] Either the executable file or the source code will be provided by the corresponding author upon request.

which there at least an equilibrium for the discriminatory auction. The non-uniqueness is a result of the discreteness of the model, and has been documented in other discrete models, as Engelbrecht-Wiggans and Kahn (1998b).[21]

We divide the rest of this section in two parts. The first part presents results on bid spread and the second on seller's revenue. Results are summarized in tables 2 to 5, that we comment along this section.

3.1 Bid Spread

This subsection illustrates how bidders use the fact that they can bid for each unit at a different price. In our model, bidders value both units on sale equally, but in most equilibria for both auction formats, they bid two different prices, i.e., there is bid spread.

Remember that given a strategy $\mathbf{s}$, for signal $x \in X$, a bidder submits bid $\mathbf{s}(x) = (s_1(x), s_2(x))$. The expected high and low bids are defined, respectively, as $E\{s_i\} := \sum_{x=0}^{K} s_i(x) \Pr(x)$, $i = 1, 2$. We define bid-spread, $\Delta(\mathbf{s})$, as the difference between the expected high and low bid, i.e., $\Delta(\mathbf{s}) := \sum_{x=0}^{K} (s_1(x) - s_2(x)) \Pr(x)$. Notice that $E\{s_1\}$, $E\{s_2\}$ and $\Delta(\mathbf{s})$ lie in $[0, 1]$. In the example in figure 2, $E\{s_1\} = 0,54$, $E\{s_2\} = 0,46$ and $\Delta(\mathbf{s}) = 0,08$.

There are several factors that explain bid spread in our model. First, bidders are risk averse, and as Wang and Zender (2002) point out, risk aversion implies diminishing marginal valuations. Risk aversion affects the bidders' strategies in two ways. First, there is standard risk premium adjustment to the bidders' marginal valuations and to their equilibrium strategy. Second, since in multiple-unit auction the risk concerns how much of the good a bidder wins, bidders will bid lower on the second unit than in the first. This effect will increase with the parameter of risk aversion, ρ.

Second, since the value of the good is common to all bidders but they have private information, bidders have to condition their bids on the information revealed by winning a particular quantity of the good: winning a large quantity is worse news about the value of the good, since it means that the others do not value the good as highly as they might. Note that this is a generalization of the Winner's curse to multiple units auctions: in one unit auctions, bidders shade their bids to avoid bidding above their conditional value for the good, but with multiple units, bidders shade their bids *below* their conditional marginal valuation, and will do so more for the second than for the first unit. This effect will increase as the number of bidders, I, increases.

Note that these two effects take place both for the Spanish and the discriminatory auction, and will be mitigated as the uncertainty about the value of the

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[21] Gordy (1996) also obtains multiple equilibria for the discriminatory auction.

good decreases because the variance of the distribution of the value of the good decreases, i.e, as α increases.

For the Spanish auction, there are two additional factors that explain bid spread and for which we expect higher bid spread for the Spanish than for the discriminatory auction. First, since in the Spanish auction there is a positive probability that the bid on the second unit will determine the price paid on the first, bidders have an incentive to shade their bid on the second unit and we expect a lower low bid, and as a consequence a higher bid spread on the Spanish than in the discriminatory equilibria. This effect will decrease as the number of bidders increases, as the mentioned probability decreases. Second, in the Spanish auction bidders have incentives to bid more aggressively on the first unit than in the discriminatory auction, since they have a lower expected cost of doing so: if they win with the highest bids, they only pay the weighted average price of winning bids instead of their bid, as they do in the discriminatory auction. In contrast to the previous effect, his effect will increase as the number of bidders increases, since bidders will increase their bids to increase the probability of winning at least one unit.

Summarizing, we expect bid spread to increase in both auction formats as the parameter of risk aversion ρ, or the uncertainty about the value of the good (the inverse of α) increases. We also expect a higher bid spread in the Spanish than in the discriminatory auction, both because bidders bid higher for the fist unit and lower for the second. As the number of bidders, I, increases, we expect bid spread to increase in the discriminatory auction, but the effect is *a priori* undetermined in the Spanish auction.

For the Spanish and the discriminatory auction, for all combinations of parameters for which there is at least an equilibrium for that auction format, Table 2 presents the average bid spread, and the high and the low expected bid across equilibria. The first column presents the average across all equilibria, and the following columns present the average of equilibria, keeping one parameter fixed, first for the number of bidders, I, next for the parameter of risk aversion, ρ, and finally for α.[22] For instance, the entry on the first row, second column (entry 1,2) in Table 2 indicates that the average bid spread across all equilibria with two bidders, $I = 2$, for the Spanish auction , is 0,052.

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[22] When there are multiple equilibria for a given combination of parameters, we calculate first the average bid spread for equilibria for that combination of parameters, and then calculate the average across parameter combinations; this procedures implies that each combination of parameters has the same weight regardless of the number of equilibria that is has, what seems natural.

For instance, consider two combinations of parameters, say A and B, such that A has 8 equilibria and B has 2. Assume for simplicity that all equilibria of A have $\Delta(\mathbf{s}) = 0.6$ and all of B have $\Delta(\mathbf{s}) = 0.2$. We first compute the average $\Delta(\mathbf{s})$ within A and B, trivially 0.6 and 0.2, respectively, and then the average across combinations: 0.4. An alternative procedure is to average directly $\frac{1}{8+2}(0.6 + ... + 0.6 + 0.2 + 0.2) = 0.52$, what gives more weight to A simply because it has more equilibria than B. We proceed similarly when calculating the expected high and low bid.

As expected, for both auction formats bid spread increases as the risk aversion parameter, ρ, or the uncertainty about the value of the good (the inverse of α) increases. For both auction formats, this is explained in most cases because the expected low bid decreases as the parameters change. Bid spread also increases for both auction formats with the number of bidders, I; the effect of increased competition is that both the low and the high bid increase, in such a way that bid spread increases.

To compare bid spread across both auction formats, we calculate, for combinations of parameter for which there is at least an equilibrium for both auction formats, the percentage of parameter combinations for which the bid spread is greater or equal for the Spanish than for the discriminatory auction, and the percentage in which average bid spread is greater for the Spanish than for the discriminatory auction. Similar information is calculated for the high and the low bid.[23] Our results confirm our *a priori* expectations: we find that for 97,4% of all combinations of parameters for which there is at least an equilibrium for both auction formats, bid spread is greater or equal in the Spanish than in the discriminatory auction. This is explained both because the high bid in the Spanish auction is greater or equal than the high bid in the discriminatory auction in 91% of the considered parameter combinations (8,3% higher on average), and because the low bid in the Spanish auction is lower or equal than the low bid in the discriminatory auction in 71,7% of the considered parameter combinations (4,4 lower on average) Table 3 summarizes this information.

3.2 Seller's Revenue

If the seller's objective is to maximize revenue, should he use the Spanish or the discriminatory format? In this subsection, we consider this question. Note that here are two opposite effects. On the one hand, as we have argued before, bidders in the Spanish auction bid more aggressively for the first unit than in the discriminatory auction, given that if they have the higher bid, they only pay the WAP; therefore, the seller's revenue could be higher for the Spanish than for the discriminatory auction. On the other hand, the fact that bidders only pay the WAP instead of their bid if they have the higher bid, and additionally, the fact that they bid lower on the second unit, suggest that the seller's revenue could be lower for the Spanish than for the discriminatory auction.

Under both auction formats, the seller's revenue depends on an I dimensional vector of observed signals, $\mathbf{x} = (x_1, ..., x_I)$, where the i-th entry is the signal observed by bidder i, and on.the equilibrium strategy played by all bidders, $\mathbf{s}$. Let $r_h(\mathbf{x}, \mathbf{s})$ be the seller's revenue given the pair $(\mathbf{x}, \mathbf{s})$ and the auction

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[23] Note that Table 2 present statistics for all equilibria, while here we only consider parameter combinations for which there is at least an equilibrium for both auctions.

format h, $h = S, D$.[24] The expected seller's revenue from the auction format h and the equilibrium strategy $\mathbf{s}$ is $R_h(\mathbf{s}) := \sum_{\mathbf{x}\in X^I} r_h(\mathbf{x},\mathbf{s}) \Pr(\mathbf{x})$. Note that $R_h(\mathbf{s}) \in [0,2]$. In the example in figure 2, $R_S(\mathbf{s}) = 1,08$. Note that if the seller is risk neutral, as it is usually assumed in the literature, he would rank both auction formats comparing expected seller's revenue.

Table 4 presents average expected seller's revenue for the Spanish and the discriminatory auction, for all equilibria, and the average for equilibria keeping one parameter constant.[25] For both auction formats average expected seller's revenue increases with the number of bidders: as competition increases, bidders increase both their bids and expected seller's revenue increases. As the parameter of risk aversion increases, average expected seller's revenue decreases in the discriminatory auction, as both the high and the low bid decrease; however, for the Spanish auction the effect in not that clear: average expected seller's revenue increases when ρ changes from 1 to 2, what is explained because the high bid increases. Finally, as the uncertainty about the value of the good decreases (α increases), for both auction formats average expected seller's revenue increases, specially for low values of α[26]. These results are in accordance with Milgrom and Weber's (1982) results for the discriminatory and uniform auction in single-unit models: bidders bid more aggressively as risk aversion decreases, and as the number of bidders of the accuracy of public information increases, and consequently the expected sellers' revenue increases.

To compare expected seller's revenue across auction formats, we consider equilibria for parameter combinations for which there is at least an equilibrium for both auction formats. The problem that arises, is how to consider expected seller's revenue for parameter combinations for which there are multiple equilibria. We propose three different alternatives, and therefore three criteria to compare expected seller´s revenue for the Spanish and the discriminatory auction.

First, we consider average expected seller's revenue, averaging across all equilibria within each combination of parameters, assuming that all occur with the same probability. Results averaging across parameter combinations are presented in the first two rows of Table 5. Expected seller's revenue is higher or equal in the Spanish than in the discriminatory auction in $64,3\%$ of the equilibria considered (on average $4,4\%$ higher). This percentage is above the average

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[24] Given the pair $(\mathbf{x},\mathbf{s})$, the i-th bidder submits $\mathbf{s}(x_i) = (s_1(x_i), s_2(x_i))$, thus the I bidders submit the aggregate vector $\mathbf{a} := (s_1(x_1), s_2(x_1), ..., s_1(x_I), s_2(x_I))$. Let $y_n(\mathbf{a})$ be the n-th order statistic of the elements of $\mathbf{a}$, that is, $(y_1(\mathbf{a}), ..., y_{2I}(\mathbf{a}))$ is a reordering of $\mathbf{a}$ satisfying $y_1(\mathbf{a}) \geq ... \geq y_{2I}(\mathbf{a})$. Then, for the Discriminatory format, $h = D$, it is $r_D(\mathbf{x},\mathbf{s}) = y_1(\mathbf{a}) + y_2(\mathbf{a})$, whereas for the Spanish format, $h = S$, it is $r_S(\mathbf{x},\mathbf{s}) = \frac{1}{2}(y_1(\mathbf{a}) + y_2(\mathbf{a})) + y_2(\mathbf{a})$, where the term $\frac{1}{2}(y_1(\mathbf{a}) + y_2(\mathbf{a}))$ is the WAP.

[25] As we did with bid spread, when there are multiple equilibria for a given parameter combination, we consider that all equilibria occur with the same probability and calculate average expected revenue accordingly.

[26] With an exception for the Spanish auction, for which average expected seller's revenue decreases when α changes from 10 to 12.

with 3 bidders $(80,9\%)$, when the parameter of risk aversion is low $(73,7\%$ and $69,2\%$ for ρ equal 1 and 2, respectively), or when the uncertainty about the value of the good is low $(67,7\%$, $71,9\%$ and 72% for α equal 12, 14 and 16, respectively).

Second, we consider the bidders' most preferred equilibrium for each combination of parameters: if there are multiple equilibria for a given combination of parameters, we select the equilibrium with the lowest seller's revenue.[27] Results averaging across parameter combinations are presented on rows three and four of Table 5. Expected seller's revenue for bidder's most preferred equilibrium is higher or equal in the Spanish than in the discriminatory auction in $69,2\%$ of the equilibria considered (on average $9,2\%$ higher). This percentage is above average in the same cases as above.

Finally, we use a stronger criterion to compare expected seller's revenue. For a given combination of parameters, we say that the Spanish auction *dominates* the discriminatory auction in terms of expected seller's revenue, if all equilibria in the Spanish auction have no lower expected seller's revenue than all equilibria in the discriminatory auction, and at least one equilibrium in the Spanish auction has higher expected seller's revenue than any in the discriminatory auction. Similarly, we define when the discriminatory auction *dominates* the Spanish auction. Results are presented on rows five and six of Table 5. According to this definition, the Spanish auction dominates the discriminatory auction in terms of expected seller's revenue in $26,7\%$ of the parameter combinations considered, and the discriminatory auction dominates the Spanish auction in $23,2\%$ (in the remaining cases, they are not comparable). When there are 3 biders, the Spanish auction dominates the discriminatory auction in $43,6\%$ of the combination of parameters considered, and the discriminatory auction dominates the Spanish only in $16,5\%$. The intuition is simple. As we explained above, bidders bid more aggressively for the first unit in the Spanish auction than in the discriminatory auction, and this effect increases with competition. This more aggressive bidding gives the seller higher expected revenue in the Spanish than in the discriminatory auction. The change in dominance when the other parameters change is not clear.

Summarizing, we have compared the expected seller's revenue for both the Spanish and the discriminatory auction in three different ways and arrive at the same conclusion: the comparison depends on the parameters considered. For some equilibria, the effect of the more aggressive bidding for the first unit in the Spanish auction than in the discriminatory auction, outperforms the effect of bidders only paying the WAP if they have the higher bid and of a lower bid on the second unit , and the Spanish auction has higher expected revenues than the

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[27] Wang and Zender (2002) present results on revenue comparison between the discriminatory and the uniform auction using the bidders' most preferred equilibrium of a uniform price auction (Proposition 3.6). Note that if bidders were able to colude, they would colude in that equilibrium.

discriminatory auction. The percentage of equilibria for which this is the case, both when we consider average expected sellers' revenue for each combination of parameters, and the bidders' most preferred equilibrium, is above average when competition is more intense, and when risk aversion or the uncertainty about the value of the good are low.

These results coincide with results reported in other papers. For example, Ausubel and Cramton (2002), who follow the "share auction" approach, provide examples that show that the ranking of the uniform and the discriminatory auction in terms of the expected seller's revenue depends on the parameters of the model. They provide examples with reasonable specifications of demand where the uniform auction dominates the discriminatory auction in terms of expected seller's revenue, and equally reasonable specifications where the opposite is true. With respect to the comparison of the Spanish and the discriminatory auctions, Álvarez et al. (2002), also using the "share auction" approach, report that if the coefficient of variation of supply to competitive bidders is small, the expected seller's revenue is higher for the Spanish than for the discriminatory auction; however, if the coefficient of variation is large, the ordering of the expected seller's revenue reverses.

4 Summary and Conclusions

This paper develops a discrete model for the Spanish auction, with multiple bids and for the common value with private information case, following Gordy (1996), that develops the model for the discriminatory auction. The Spanish auction is a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids, and pay the weighted average price of winning bids otherwise. We numerically find all (if any) the Bayesian Nash symmetric equilibria for a wide range of parameter values.

Our main results are the following. First, in $97,4\%$ of the combinations of parameter considered, bid spread is higher in the Spanish than in the discriminatory auction; this is explained because in 91% of the combinations of parameters considered, the high bid (the bid for the first units) is higher in the Spanish than in the discriminatory format. Second, precisely because bidders are more aggressive with their high bids in the Spanish than in the discriminatory format, the former outperforms the latter in terms of expected seller's revenue in $69,2\%$ of the combinations of parameters considered, when we choose bidders' preferred equilibrium when there are multiple equilibria. More interestingly, the percentage of equilibria for which this is the case, is above average when competition is more intense, and when risk aversion or the uncertainty about the value of the good are low.

The design of Treasury auctions is important, since countries are using them to finance public debts. We think that our results suggest that Treasuries around the world should pay attention to the Spanish auction format. Opponents of the discriminatory auction, the most widely used auction format, argue that bidders shade their bids downward more in the discriminatory than in the uniform format, aware of the cost of overbidding, and that as a result, seller's revenue is lower. For example, the U.S. Treasury, when arguing in favor of the use of uniform auctions, the format that it is now using, said that "auction participants may bid more aggressively in single-price auctions. Successful bidders are able to avoid the so-called "winner's curse". (...) We estimate that more aggressive bidding has lowered Treasury borrowing cost somewhat".[28] However, the U.S. Treasury experiment yielded inconclusive results: "In a direct comparison of the impact on revenue between the two techniques, the data show a small increase in revenues to the Treasury under the uniform-price technique, but the different is not statistically significant".[29] The Spanish auction mitigates the downward bias on bidding with respect to the discriminatory format, and could increase participation. Our results show that the expected seller's revenue is higher than with discriminatory auctions in a number of cases. If Treasuries are changing from discriminatory to uniform auctions, why not try a hybrid of both that presents good properties?

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Appendix

1. Evaluating the expected utility of a strategy

In this appendix, we present some basic results to calculate the expected utility for bidder i of bidding an arbitrary strategy $\widehat{\mathbf{s}}$, given that all other bidders are bidding $\mathbf{s}_{-i} = (\mathbf{s}, ..., \mathbf{s})$, conditional on signal x_i. Let Ω be the set of events that might occur to a bidder under the Spanish format, and let σ be an element in Ω. The elements in Ω are: (a) to get two units, and pay the WAP for the first unit and the low bid on $\widehat{\mathbf{s}}$ for the second unit; (b) to get one unit and pay the high bid on $\widehat{\mathbf{s}}$; (c) to get one unit and pay the WAP; and (d) to get zero units. We can write:

$$E\left\{U_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}\left(\mathbf{x}_{-i}\right)\right) \ / \ x_i\right\} = \sum_{\sigma \in \Omega} E\left\{U_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}\left(\mathbf{x}_{-i}\right)\right) \ / \ x_i, \sigma\right\} \Pr\left(\sigma \ / \ \widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i\right)$$

where $\Pr\left(\sigma \ / \ \widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i\right)$ is the probability of event σ conditional on $\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i\right)$ and $\mathbf{x}_{-i} \in X^{I-1}$ is the $I-1$-th dimensional vector of signals that the rivals (of bidder i) observe.

Proposition 1 presents some simplifications to compute each of the terms in the previous summation. First, let us denote each of those terms as follows:
$V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, \sigma\right) := E\left\{U_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}\left(\mathbf{x}_{-i}\right)\right) \ / \ x_i, \sigma\right\} \Pr\left(\sigma \ / \ \widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i\right)$
and $\Omega := \{a, b, c, d\}$, where the elements in Ω are as listed above.

Proposition 1. *For the Spanish format, the following holds:*
(i) $V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, a\right) = -\sum_{\mathbf{x}_{-i}} p^a\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right) z^a\left(\mathbf{x}_{-i}, x_i\right)$
(ii) $V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, b\right) = -\exp\left(\rho\left(1 - \widehat{s}^1\right)\right) \sum_{\mathbf{x}_{-i}} p^b\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right) z^b\left(\mathbf{x}_{-i}, x_i\right)$
(iii) $V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, c\right) = -\sum_{\overline{s}} \exp\left(\rho\left(1 - \frac{1}{2}\left(\widehat{s}^1 + \overline{s}\right)\right)\right) \sum_{\mathbf{x}_{-i}} p^c\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right) z^c\left(\mathbf{x}_{-i}, x_i\right)$
(iv) $V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, d\right) = -\exp\left(\rho\left(2 - \frac{3}{2}\widehat{s}^1 - \frac{1}{2}\widehat{s}^2\right)\right) \sum_{\mathbf{x}_{-i}} p^d\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right) z^d\left(\mathbf{x}_{-i}, x_i\right)$
where $p^\sigma\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right)$ *denotes the probability that event* σ *takes place conditional on* $\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right)$ *and* $z^\sigma\left(\mathbf{x}_{-i}, x_i\right)$ *is defined as follows:*
$z^a\left(\mathbf{x}_{-i}, x_i\right) = g_{\mathbf{x}_{-i}}\left(\mathbf{x}_{-i} \ / \ x_i\right)$;
$z^b\left(\mathbf{x}_{-i}, x_i\right) =_1 F_1\left(IK - \left(\mathbf{1}^T\mathbf{x}_{-i} + x_i\right) + \alpha\left(1-\mu\right), IK + \alpha, \rho\right) g_{\mathbf{x}_{-i}}\left(\mathbf{x}_{-i} \ / \ x_i\right)$;
$z^c\left(\mathbf{x}_{-i}, x_i\right) = z^b\left(\mathbf{x}_{-i}, x_i\right)$;
$z^d\left(\mathbf{x}_{-i}, x_i\right) =_1 F_1\left(IK - \left(\mathbf{1}^T\mathbf{x}_{-i} + x_i\right) + \alpha\left(1-\mu\right), IK + \alpha, 2\rho\right) g_{\mathbf{x}_{-i}}\left(\mathbf{x}_{-i} \ / \ x_i\right)$,
where $1^T x_{-i}$ *is the summation of the components of* x_{-i}*,* $g_{\mathbf{x}_{-i}}\left(\mathbf{x}_{-i}/x_i\right)$ *is the density of* x_{-i} *conditional on* x_i *and* ${}_1F_1\left(.\right)$ *is the confluent hypergeometric function (see Abramowitz and Stegun (1972)).*

This proposition is straightforward from Gordy (1996), and thus we omit the proof. The interest of the proposition is to decouple, for instance, $V_i\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, x_i, b\right)$ in *(ii)*, into two terms. The first, $-\exp\left(\rho\left(1 - \widehat{s}^1\right)\right)$, depends only on $\widehat{\mathbf{s}}$. The second term is a summation across the possible vectors of rivals' signals, $\mathbf{x}_{-i}$, which is a summation of a finite number of terms. Furthermore, each term in this summation is a product: $p^b\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right) z^b\left(\mathbf{x}_{-i}, x_i\right)$, where $p^b\left(\widehat{\mathbf{s}}, \mathbf{s}_{-i}, \mathbf{x}_{-i}, x_i\right)$ is easily computable and $z^b\left(\mathbf{x}_{-i}, x_i\right)$ does not depend on strategies, and thus only has to be computed once for every vector of parameter values.

Additionally, it can be proven that:
$g_{\mathbf{x}_{-i}}\left(\mathbf{x}_{-i} \;/\; x_i\right) = M\left(\mathbf{x}_{-i}\right) \Pi_{i=1}^{I-1} \binom{K}{\mathbf{x}_{-i}(j)} \frac{B(x_i+\alpha\mu, K-x_i+\alpha(1-\mu))}{B(\alpha\mu, \alpha(1-\mu))}$
where $M(.)$ and $B(.)$ are the multinomial and beta functions, respectively (see DeGroot (1970)), and $\mathbf{x}_{-i}(j)$ is the j-th entry of $\mathbf{x}_{-i}$.

2. Figures

Figure 1.

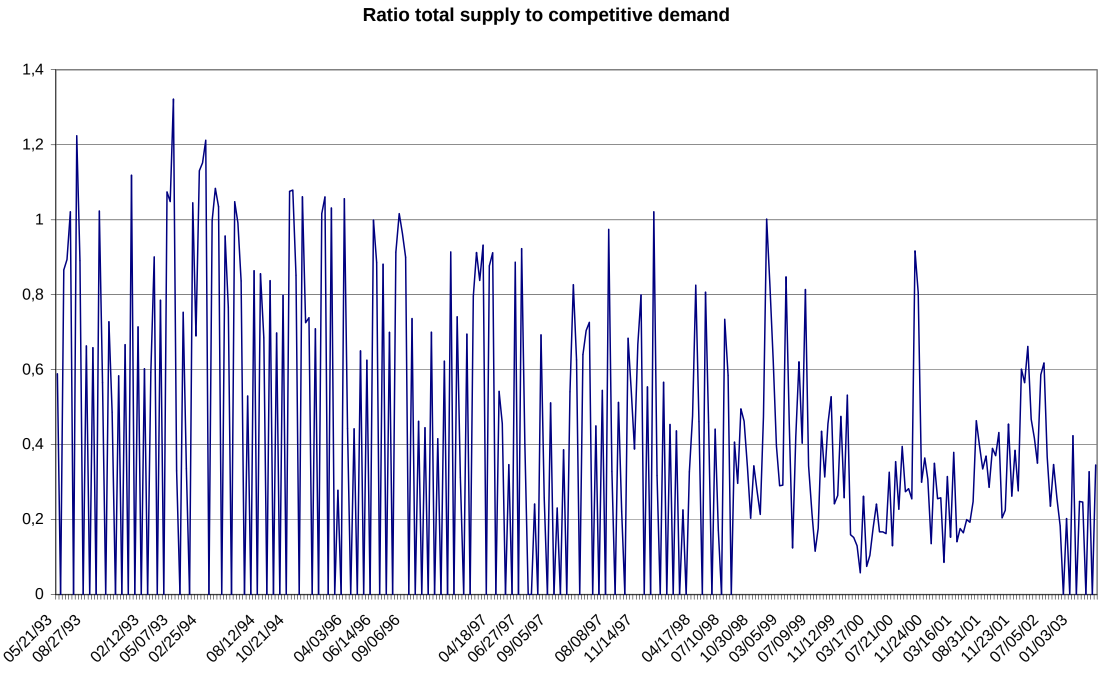
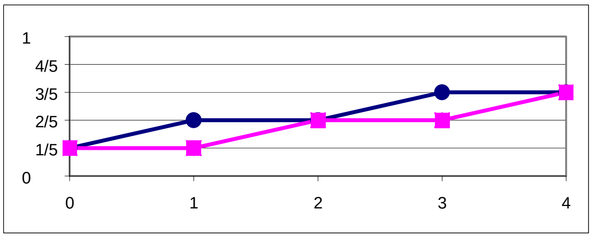


Figure 2. An example of equilibrium under the Spanish format



3. Tables

Table 1: Computation time

$I = 2$					
		K			
		3	4	5	6
λ	3	1	1	1	1
	4	1	1	4	8
	5	1	3	8	220
	6	1	9	122	—
$I = 3$					
		K			
		3	4	5	6
λ	3	1	1	2	2
	4	2	2	6	117
	5	1	10	99	909
	6	3	34	458	—

The table shows the computation time rounded to minutes. For instance, the entry corresponding to $(I, \lambda, K) = (2, 3, 3)$ indicates that, for each combination of parameters containing those values (regardless the value of any other parameter), it takes 1 minute to check all possible strategies for both formats.